MLQ

MATHEMATICAL LOGIC QUARTERLY

A Journal for Mathematical Logic, Foundations of Mathematics, and Logical Aspects of Theoretical Computer Science

www.mlq-journal.org

Published under the auspices of the

DVMLG

Deutsche Vereinigung für Mathematische Logik und für Grundlagenforschung der Exakten Wissenschaften

WILEY-VCH

Math. Log. Quart. 1–9 (2017) / **DOI** 10.1002/malq.201500071

Killing them softly: degrees of inaccessible and Mahlo cardinals

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Received 7 October 2015, revised 24 June 2016, accepted 25 July 2016
Published online 31 October 2017

This paper introduces the theme of killing-them-softly between set-theoretic universes. The main theorems show how to force to reduce the large cardinal strength of a cardinal to a specified desired degree. The killing-them-softly theme is about both forcing and the gradations in large cardinal strength. Thus, I also develop meta-ordinal extensions of the hyper-inaccessible and hyper-Mahlo degrees. This paper extends the work of Mahlo to create new large cardinals and also follows the larger theme of exploring interactions between large cardinals and forcing central to modern set theory.

1 Introduction

This paper is, in equal parts, about degrees of large cardinals and about forcing to reduce the degree of a large cardinal. The focus here are the inaccessible and Mahlo cardinals, relatively small large cardinals in the large cardinal hierarchy. In the early 20th century, Mahlo found a way to generate new large cardinals [6]. In fact, he found a way to generate an entire hierarchy of large cardinals which fits between Mahlo and weakly compact. I was interested in doing this between inaccessible and Mahlo. Set-theorists are already aware of the hierarchy of degrees of inaccessible cardinals, although it is not explicitly described in the literature beyond hyper-inaccessible. I provide here explicit definitions (including new words) for the infinite levels of degrees of inaccessible cardinals. Undertaking this project, one realizes that a notation system for this infinite hierarchy must go beyond the ordinals. For this reason, I use meta-ordinals to denote the degrees of inaccessible cardinals. We use the symbol Ω as a formal syntactic expression for the order type of Ord. We shall see how one can use meta-ordinals to describe this hierarchy. We were able to do for inaccessible cardinals what Mahlo did for Mahlo cardinals, and even beyond what he did. Thus, also included in this paper are formal definitions using meta-ordinals for higher degrees of Mahlo cardinals.

The motivation for the main theorems of this paper was the desire to force to reduce the large cardinal strength of a cardinal by one degree. In other words, I wanted to travel to a universe where the strength of a given large cardinal dies, but just a little. There are potentially infinitely many killing-them-softly theorems like the ones included here. The main theorems in this paper are about softly killing large cardinal strength for degrees of inaccessible and Mahlo cardinals. In a forthcoming paper, I shall do the same for degrees of measurable and supercompact cardinals.

2 Degrees of inaccessible cardinals

An inaccessible cardinal κ is an uncountable cardinal which is a regular strong limit cardinal. A cardinal κ is α-inaccessible if and only if κ is inaccessible and for every $\beta < \alpha$, κ is a limit of β-inaccessible cardinals.

The following is the first killing-them-softly result and the main theorem of this section.

Theorem 2.1 *If κ is α-inaccessible, then there is a forcing extension where κ is still α-inaccessible, but not $(\alpha + 1)$-inaccessible.*

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Before the proof of this theorem, is the following Lemma which establishes some basic facts about degrees of inaccessible cardinals.

Lemma 2.2 *(1) A cardinal κ is 0-inaccessible if and only if κ is inaccessible. (2) If κ is α-inaccessible, and $\beta < \alpha$, then κ is also β-inaccessible. (3) A cardinal κ cannot be η-inaccessible, for any $\eta > \kappa$.*

Proof. The proofs of parts (1) and (2) are trivial. For (3), by way of contradiction, suppose κ is the least cardinal with the property that κ is $(\kappa + 1)$-inaccessible. It follows from the definition that κ is a limit of κ-inaccessible cardinals. Thus, there is a cardinal $\beta < \kappa$ for which β is κ-inaccessible. Since κ is a limit ordinal, $\beta + 1 < \kappa$, and so by statement (2), β is also $(\beta + 1)$-inaccessible. This contradicts that κ is the least cardinal with this property. It follows from (2) that κ cannot be η-inaccessible for any $\eta \geq \kappa + 1$. □

And now we proceed to the proof of Theorem 2.1:

Proof. Let κ be α-inaccessible. If κ is not $(\alpha + 1)$-inaccessible, then trivial forcing will give the forcing extension where κ is α-inaccessible, but not $(\alpha + 1)$-inaccessible. Thus, assume that κ is $(\alpha + 1)$-inaccessible, and we shall find a forcing extension where it is no longer $(\alpha + 1)$-inaccessible but still α-inaccessible. The idea of the proof is to add a club, C, to κ which contains no α-inaccessible cardinals, and then force to change the continuum function to kill strong limits which are not limit points of C. To change the continuum function, we shall perform Easton forcing.

Easton forcing to change the continuum function works when the GCH holds in the ground model. For our purposes, we only need the GCH pattern to hold up through κ, by forcing with $\mathbb{P}$, which is an Easton support iteration of length κ of $\mathrm{Add}(\gamma, 1)$ for regular $\gamma \in V^{\mathbb{P}_\gamma}$. The forcing $\mathbb{P}$, neither destroys, nor creates, inaccessible cardinals below κ. Let $G \subseteq \mathbb{P}$ be V-generic. Then V and $V[G]$ have the same β-inaccessible cardinals for any $\beta < \kappa$, and further $V[G] \models \mathsf{GCH}$.

Next, force with $\mathbb{C}$, which will add a club subset to κ, which contains no α-inaccessible cardinals in the forcing extension. Conditions $c \in \mathbb{C}$ are closed, bounded, subsets of κ, consisting of infinite cardinals and containing no α-inaccessible cardinals. The forcing $\mathbb{C}$ is ordered by end-extension: $d \leq c$ if and only if $c = d \cap (\sup(c) + 1)$. Let $H \subseteq \mathbb{C}$ be $V[G]$-generic, and let $C = \bigcup H$. Then, clearly C is a club in κ, and it contains no α-inaccessible cardinals.

We have seen that the forcing, $\mathbb{C}$, adds a new club, C, to κ. It remains to show that C contains unboundedly many β-inaccessible cardinals, and that $\mathbb{C}$ preserves cardinals, cofinalities and strong limits. The new club, C, does not contain any ground model α-inaccessible cardinals. The following argument shows that the new club, C, contains unboundedly many ground model β-inaccessible cardinals, for every $\beta < \alpha$. Fix $\beta < \alpha$ and $\eta < \kappa$. Let D_η be the set of conditions in $\mathbb{C}$ which contain a β-inaccessible γ above η, and which contain a sequence of inaccessible cardinals, unbounded in γ, witnessing that γ is β-inaccessible. Let us see that D_η is dense in $\mathbb{C}$. Let $c \in \mathbb{C}$, and let γ be the next β-inaccessible above both η and and the maximal element of c. Since γ is the next β-inaccessible past η and $\beta < \alpha$, there are no α-inaccessible cardinals in $(\eta, \gamma]$. Also, this block, $(\eta, \gamma]$, contains the tails of all sequences of inaccessible cardinals, which witnesses that γ is β-inaccessible. Let $d = c \cup ((\eta, \gamma] \cap \mathrm{Card})$. Then, d extends c, contains a β-inaccessible above η, and a sequence of inaccessible cardinals which witness that γ is β-inaccessible. Thus, $d \in D_\eta$, which shows that D_η is dense in $\mathbb{C}$, and thus shows that C contains unboundedly many β-inaccessible cardinals. From this fact also follows, that, in the final extension, κ is still α-inaccessible. Finally, the forcing $\mathbb{C}$ preserves cardinals and cofinalities greater than or equal to $\kappa + 1$, forcing over $V[G] \models \mathsf{GCH}$, since $|\mathbb{C}| = \kappa^{<\kappa} = \kappa$.

For $\beta < \kappa$, the forcing, $\mathbb{C}$, is not $\leq\beta$-closed, since if β is α-inaccessible, there is a β-sequence of conditions unbounded in β, but no condition could close the sequence since it would have to include β. However, for every $\beta < \kappa$, the set $D_\beta = \{d \in \mathbb{C} \,:\, \max(d) \geq \beta\}$ is dense in $\mathbb{C}$ and is $\leq\beta$-closed. This is true, since for any β-sequence of conditions in D_β, one can close the sequence by taking unions at limits and adding the top point, which cannot be inaccessible, because it is not regular, since this top point is above β, but has cofinality β. Thus, for every $\beta < \kappa$, the forcing, $\mathbb{C}$, is forcing equivalent to D_β, which is $\leq \beta$-closed. Thus, $\mathbb{C}$ preserves all cardinals, cofinalities, and strong limits. Thus. $V[G][H] \models \mathsf{GCH}$ and we have forced to add $C \subseteq \kappa$, club, which contains no α-inaccessible cardinals.

The last step of the proof is to force over $V[G][H]$, with $\mathbb{E}$, Easton's forcing to change the continuum function. Specifically, let $\mathbb{E}$ force $2^{\gamma^+} = \delta^+$, where $\gamma \in C$ and δ is the next element of C past γ^+. This forcing preserves all cardinals and cofinalities [3, pp. 232ff.], and also preserves that κ is inaccessible. However, $\mathbb{E}$ does not preserve all inaccessible cardinals below κ. In fact, $\mathbb{E}$ destroys all strong limits which are not limit points of C. Let $K \subseteq \mathbb{E}$ be $V[G][H]$-generic. Since C', the set of limit points of C, contains no ground model α-inaccessible cardinals, there are no α-inaccessible cardinals below κ in $V[G][H][K]$. Thus, κ is not $(\alpha + 1)$-inaccessible in the final forcing extension.

Finally since C' contains unboundedly many ground model β-inaccessible cardinals (which is the same as the set of β-inaccessible cardinals in the intermediate extensions since $\mathbb{P}$ and $\mathbb{C}$ preserve all inaccessible cardinals) for every $\beta < \alpha$. The cardinal κ is still β-inaccessible in $V[G][H][K]$. Thus, in $V[G][H][K]$, for every $\beta < \alpha$, there are unboundedly many β-inaccessible cardinals below κ. Thus, κ is still α-inaccessible in the final extension. But, κ is not $(\alpha + 1)$-inaccessible in $V[G][H][K]$ since C' contains no α-inaccessible cardinals. □

If a cardinal κ is κ-inaccessible, then it is defined to be *hyper-inaccessible*. Therefore, if κ is hyper-inaccessible, Theorem 2.1 shows how to force to make κ have inaccessible degree exactly α for any fixed $\alpha < \kappa$. One can force to change a hyper-inaccessible cardinal to have maximal degree by forcing to add a club which avoids all degrees above the desired degree, and then force with Easton forcing to destroy strong limits which are not limit points of the new club, as in the proof of Theorem 2.1. I'll state it as a corollary below.

Corollary 2.3 *If κ is hyper-inaccessible, then for any $\alpha < \kappa$, there exists a forcing extension where κ is α-inaccessible, but not hyper-inaccessible.*

Lemma 2.2 shows that the greatest degree of α-inaccessibility that κ can be is κ-inaccessible. Mahlo began the investigation of degrees of inaccessible cardinals [5], and fully defined the analgous notions for Mahlo cardinals [6], and I shall continue his work by formalizing the degrees of inaccessible cardinals in the remainder of this section. So, how do we proceed beyond hyper-inaccessible cardinals in defining degrees of inaccessibility? By repeating the process: a cardinal κ is *1-hyper-inaccessible* if and only if κ is hyper-inaccessible, and a limit of hyper-inaccessible cardinals (0-hyper-inaccessible is hyper-inaccessible). That is, κ is 1-hyper-inaccessible if and only if the set $\{\gamma < \kappa \ : \ \gamma \text{ is } \gamma\text{-inaccessible}\}$ is unbounded in κ. A cardinal κ is *α-hyper-inaccessible* if and only if κ is hyper-inaccessible, and for every $\beta < \alpha$, the cardinal κ is a limit of β-hyper-inaccessible cardinals. A cardinal κ is *hyper-hyper-inaccessible*, denoted hyper^2-inaccessible, if and only if κ is κ-hyper-inaccessible (hyper^0-inaccessible denotes inaccessible). The following theorem shows we have again reached an apparent roadblock in the hierarchy of inaccessible cardinals.

Theorem 2.4 *If κ is α-hyper-inaccessible, then $\alpha \leq \kappa$.*

The proof is similar to the proof of Lemma 2.2.

In order to reach higher degrees of inaccessibility past this limit, repeat the process: a cardinal κ is 1-hyper^2-*inaccessible* if and only if κ inaccessible and a limit of hyper^2-inaccessible cardinals. A cardinal κ is α-hyper^2-*inaccessible* if and only if, for every $\beta < \alpha$, the cardinal κ is inaccessible and a limit of β-hyper^2-inaccessible cardinals. By the same argument as in Lemma 2.2, α must be less than or equal to κ in this definition. And, similarly, a cardinal κ is *hyper^3-inaccessible* if and only if κ is κ-hyper^2-inaccessible.

Definition 2.5 A cardinal κ is α-hyper$^\beta$-*inaccessible* if and only if

(1) the cardinal κ is inaccessible, and
(2) for all $\eta < \beta$, the cardinal κ is κ-hyper$^\eta$-inaccessible, and
(3) for all $\gamma < \alpha$, the cardinal κ is a limit of γ-hyper$^\beta$-inaccessible cardinals.

Definition 2.5 subsumes the previous definitions of hyper-inaccessible cardinals. For example, 0-hyper^0-inaccessible is just inaccessible, since the second and third parts of the definition are not applicable. If κ is α-hyper^0-inaccessible, then by definition, κ is inaccessible and for all $\gamma < \alpha$, the cardinal κ is a limit of γ-hyper^0-inaccessible cardinals, hence κ is α-inaccessible. And if κ is α-inaccessible, then it is inaccessible and a limit of β-inaccessible cardinals, for every $\beta < \alpha$, hence κ is α-hyper^0-inaccessible. And, 0-hyper-inaccessible is just hyper-inaccessible since, from the general definition, this means that κ is inaccessible and for every $\eta < 1$, the cardinal κ is κ-hyper$^\eta$-inaccessible, i.e., κ is κ-hyper^0-inaccessible, hence hyper-inaccessible. In the previous definition of hyper-inaccessible, it was required that κ be hyper-inaccessible, but this requirement is included in the second part of Definition 2.5 since this implies that κ is hyper-inaccessible whenever $\beta > 0$.

Definition 2.5 gives a general definition for hyper-inaccessible cardinals. The following theorem shows that a cardinal κ can be at most hyper$^\kappa$-inaccessible, using this definition.

Theorem 2.6 *A cardinal κ cannot be 1-hyper$^\kappa$-inaccessible.*

The proof is similar to the proof of Lemma 2.2.

Theorem 2.6 shows that in order to continue defining the higher degrees of inaccessible cardinals, we need to relativize, just as was done before. That is, while a cardinal κ cannot be $(\kappa + 1)$-inaccessible, a limit of κ-inaccessible cardinals, it can be 1-hyper-inaccessible, a limit of hyper-inaccessible cardinals. Here, the word hyper allows us to define a higher degree. We are in exactly the same situation now. Theorem 2.6 shows that a cardinal κ cannot be 1-hyper$^\kappa$-inaccessible, a limit of hyper$^\kappa$-inaccessible cardinals. Therefore, to define the higher degrees of inaccessible cardinals, as in Definition 2.5, we need more words.

In order for us to move forward, let us say that a cardinal κ is *richly-inaccessible* if and only if κ is hyper$^\kappa$-inaccessible. The word richly expresses that such a cardinal is brimming with inaccessible cardinals. Now we can get past the roadblock of Theorem 2.6 because a cardinal κ can be 1-richly-inaccessible, a limit of richly-inaccessible cardinals, and we can go forever. Each time we get stuck, we just need a new word. Therefore, I have assigned the first few words to describe the higher degrees of inaccessible cardinals, so that we can keep going.

Before I give the words for the higher degrees, I should like to mention that we do not need a new word each time we are apparently stuck (and need to relativize). We can re-use the words we already have to be as efficient as possible. Suppose a cardinal κ is κ-richly-inaccessible. Then we can say that κ is hyper-richly-inaccessible. We know that κ cannot be $(\kappa + 1)$-richly-inaccessible, so we are a little stuck if we want to keep going. But, a cardinal κ can be 1-hyper-richly-inaccessible, so we can continue. We really only need a brand new word when we reach an obstacle like in Theorem 2.6.

Below are the new words to describe the higher degrees of inaccessible cardinals, chosen to express qualities of the Infinite: A cardinal κ is *utterly-inaccessible* if and only if κ is richly$^\kappa$-inaccessible; it is *deeply-inaccessible* if and only if κ is utterly$^\kappa$-inaccessible; it is *truly-inaccessible* if and only if κ is deeply$^\kappa$-inaccessible; it is *eternally-inaccessible* if and only if κ is truly$^\kappa$-inaccessible; and it is *vastly-inaccessible* if and only if κ is eternally$^\kappa$-inaccessible.

I should like to give a couple more examples to show how we can re-use words, and use exponents to express the higher degrees. A cardinal κ is hyper^2-richly-inaccessible if and only if κ is κ-hyper-richly-inaccessible. A cardinal κ is richly-utterly-inaccessible if and only if κ is hyper$^\kappa$-utterly-inaccessible. Also, I should like to be clear that any combination of the words makes senses; a cardinal could even be hyper-richly-utterly-deeply-truly-eternally-vastly-inaccessible.

Recall that the diagonal intersection of a collection of classes is defined as $\triangle_{\alpha\in I} X_\alpha = \{\kappa \;:\; \kappa \in \bigcap_{\alpha\in I\cap\kappa} X_\alpha\}$, for some index class I. The words hyper, richly, utterly, and so on mark places in the process of defining degrees of inaccessible cardinals where we take a diagonal intersection over of a collection of classes, where the index class is Ord. For example, the class of hyper-inaccessible cardinals is the diagonal intersection of the classes of α-inaccessible cardinals over all $\alpha \in$ Ord. And, the class of richly-inaccessible is the diagonal intersection of the collection of hyper$^\alpha$-inaccessible cardinals over all $\alpha \in$ Ord. Thus, our notation system for describing the classes of degrees of inaccessible cardinals uses meta-ordinals. A meta-ordinal is a formal syntactic expression for order-types beyond and including Ord. We use the symbol Ω for the order type of Ord. Then any meta-ordinal can be expressed in terms of Ω, just like the role ω plays in Cantor's normal form for ordinals. Let us only consider meta-ordinals of the form $\Omega^\alpha \cdot \beta + \Omega^\eta \cdot \gamma + \cdots + \Omega \cdot \delta + \sigma$ where the exponents and coefficients are ordinals. We require that the normal form of a meta-ordinal has the terms from left to right in decreasing order according to the exponents.

If s and t are meta-ordinals then the ordering is lexicographical. If the degree on Ω of the leading term of t is greater than s, then $s < t$. If s and t have the same greatest degree of Ω, then compare the coefficients of the leading term. If these are the same, compare the next highest powers of Ω in s and t and so on.

Now that we have these meta-ordinals, we can describe the classes of degrees of inaccessible cardinals in a uniform way. If η and κ are hyper-inaccessible cardinals then η is in the class of η-inaccessible cardinals and κ is in the class of κ-inaccessible cardinals. But now I can describe both as being in the class of Ω-inaccessible cardinals, which is the diagonal intersection of all classes of α-inaccessible cardinals, for $\alpha \in$ Ord. Defined in this way, the degree of an inaccessible cardinal κ can be described as t-inaccessible for some meta-ordinal t. The only restriction being that all of the ordinals in t are less than or equal to κ. In this way, inaccessible cardinals with the same degree of inaccessibility can be described with the same meta-ordinal.

Definition 2.7 If t is a meta-ordinal with ordinal terms less than κ, then a cardinal κ is t-inaccessible if and only if κ is inaccessible and for every meta-ordinal $s < t$ having ordinal terms less than κ, the cardinal κ is a limit of s-inaccessible cardinals.

This definition of t-inaccessibility for a cardinal κ can be extended to include meta-ordinal terms t with just ordinal terms less than or equal to κ by replacing κ by Ω. For example, a cardinal κ which is κ-inaccessible is now called Ω-inaccessible. A cardinal κ which is $\Omega \cdot \kappa$-inaccessible is a hyper$^\kappa$-inaccessible cardinal which is called a richly-inaccessible cardinal, or an Ω^2-inaccessible cardinal. Thus,

$$\begin{aligned}
\kappa \text{ is } \Omega\text{-inaccessible} &\quad \text{if and only if } \kappa \text{ is hyper-inaccessible,}\\
\kappa \text{ is } \Omega^2\text{-inaccessible} &\quad \text{if and only if } \kappa \text{ is richly-inaccessible,}\\
\kappa \text{ is } \Omega^2+\Omega\text{-inaccessible} &\quad \text{if and only if } \kappa \text{ is hyper-richly-inaccessible,}\\
\kappa \text{ is } \Omega^3\text{-inaccessible} &\quad \text{if and only if } \kappa \text{ is utterly-inaccessible,}\\
\kappa \text{ is } \Omega^3+\Omega^2\text{-inaccessible} &\quad \text{if and only if } \kappa \text{ is richly-utterly-inaccessible,}\\
\kappa \text{ is } \Omega^4\text{-inaccessible} &\quad \text{if and only if } \kappa \text{ is deeply-inaccessible,}\\
\kappa \text{ is } \Omega^5\text{-inaccessible} &\quad \text{if and only if } \kappa \text{ is truly-inaccessible,}\\
\kappa \text{ is } \Omega^6\text{-inaccessible} &\quad \text{if and only if } \kappa \text{ is eternally-inaccessible, and}\\
\kappa \text{ is } \Omega^7\text{-inaccessible} &\quad \text{if and only if } \kappa \text{ is vastly-inaccessible.}
\end{aligned}$$

Thus κ is $(\Omega^7+\Omega^6+\Omega^5+\Omega^4+\Omega^3+\Omega^2+\Omega+\alpha)$-inaccessible if and only if κ is α-hyper-richly-utterly-deeply-truly-eternally-vastly-inaccessible. Now that we have this uniform notation, we can softly kill any successor inaccessible degree.

Theorem 2.8 *If κ is t-inaccessible, then there is a forcing extension where κ is still t-inaccessible, but not $(t+1)$-inaccessible.*

Proof Sketch. The proof is very similar to the proof of Theorem 2.1. Assume $V \models$ ZFC. Suppose $\kappa \in V$ is t-inaccessible, where t is a meta-ordinal term with only ordinal terms less than κ. Force with $\mathbb{C}$, the club shooting forcing, as before. Conditions are closed, bounded subsets of κ such that if $c \in \mathbb{C}$ then c contains no t-inaccessible cardinals. The forcing $\mathbb{C}$ preserves cardinals and cofinalities. Since $\forall \gamma < \kappa$, the set $D_\gamma = \{d \in \mathbb{C} : \max(d) > \gamma\}$ is dense, the new club $C = \bigcup G \subseteq \mathbb{C}$ is unbounded, where G is V-generic. It follows that C is closed, since it is unbounded and the conditions are closed, as before. For every $s < t$, where s is a meta-ordinal term with ordinal parameters less than κ, the set $D_s = \{d \in \mathbb{C} \ : \ d$ contains a block of cardinals up to an s-inaccessible cardinal$\}$ is dense: if $c \in \mathbb{C}$, then if η is the least s-inaccessible cardinal above η, then $d = c \cup ([\max(c), \eta] \cap \mathrm{Card})$ is in D_s.

Next force with $\mathbb{E}$: For infinite $\gamma \in C$, force $2^{\gamma^+} = \delta^+$ where δ is the next element of C past γ^+. $\mathbb{E}$ destroys strong limits which are not in C' and if $\eta \in C'$ is a strong limit, then η remains a strong limit in the final extension.

Since C contains unboundedly many ground model s-inaccessible cardinals for $s < t$, with a block of cardinals below them, then in the final extension these cardinals are still s-inaccessible (by a proof by meta-ordinal induction on s), and so κ is still t-inaccessible. However since all t-inaccessible cardinals below κ are no longer strong limits, κ is not $t+1$-inaccessible. □

3 Mahlo cardinals

This section begins with theorems about Mahlo cardinals, showing that Mahlo cardinals have all the inaccessible large cardinal properties from the last section. Also, an analogue of the classical notion of greatly Mahlo is defined for inaccessible cardinals. The classical degrees of Mahlo cardinals are also described, and theorems which softly kill them. The referee for this paper was very helpful to point out that there has been work done with Mahlo cardinals which is similar to the theme here. One of the earlier results of this kind is Jensen's theorem for turning a Mahlo cardinal into an inaccessible non-Mahlo by shooting a club of singular cardinals through it [4]. William Boos has extended Jensen's result for degrees of Mahlo cardinals, in particular the main theorem of § 3, i.e., Theorem 14 is proved by him in Boolean extensions which efface the Mahlo [1]. The author should mention this fact. There is also similar result about subtle cardinals by Claudia Henrion in "Properties of subtle cardinals" [2].

An inaccessible cardinal κ is *Mahlo* if and only if the set of inaccessible cardinals below κ is a stationary subset of κ. A cardinal κ is *greatly inaccessible* if and only if there is a uniform, normal filter on κ, closed under the inaccessible limit point operator:

$$\mathcal{I}(X) = \{\alpha \in X \ : \ \alpha \text{ is an inaccessible limit point of } X\}.$$

A uniform filter on κ has for every $\beta < \kappa$, the set $[\beta, \kappa)$ is in the filter, and normal means that the filter is closed under diagonal intersections $\Delta_{\alpha<\kappa}$. The last part of the definition means that if X is in the filter, so is $\mathcal{I}(X)$. The first theorem of this section shows that greatly inaccessible is equivalent to Mahlo.

Theorem 3.1 *A cardinal κ is greatly inaccessible if and only if κ is Mahlo.*

Proof. For the forward implication, suppose κ is Mahlo. Let F be the filter generated by sets of the form $C \cap I$, where C is club in κ, and I is the set of inaccessible cardinals below κ. Note that κ is club in itself, and $\kappa \cap I = I$, so $I \in F$. Then, the claim is that F is a uniform, normal filter, closed under $\mathcal{I}$. First, if C, D are club in κ, then $C \cap D$ is club. Thus, for any clubs C and D, the equation $(C \cap I) \cap (D \cap I) = (C \cap D) \cap I$ implies that any set A in the filter, generated by sets of the form $C \cap I$, where C is club, is itself a superset of a set of the form $E \cap I$, where E is club in κ. Then, $\varnothing \notin F$, since the empty set has no non-trivial subsets, hence cannot be a superset of the form $C \cap I$. Next, if $A \in F$, and $A \subseteq B$, then there is a club $C \in F$ such that $C \cap I \subseteq A \subseteq B$, thus $B \in F$, by construction, since F is the filter generated by sets of this form. Third, if A and B are elements of F, then there are clubs C and D such that $C \cap I \subseteq A$, and $D \cap I \subseteq B$, so $A \cap B$ contains $(C \cap D) \cap I$. This is of the form which generated the filter, thus $A \cap B \in F$. Also, F is uniform since if b is a bounded subset of κ, then $\kappa \setminus b$ contains a club E. Thus, $E \cap I \subseteq E \subseteq \kappa \setminus b$, so $\kappa \setminus b \in F$. Since the cardinality of any co-bounded set is κ, the filter is uniform. It remains to show that F is normal, and closed under the inaccessible limit point operation. The fact that F is normal, follows easily from the fact that clubs are closed under diagonal intersection.

Finally, for this direction, to show Mahlo implies greatly inaccessible, F is closed under $\mathcal{I}$, since if $A \in F$, then $\mathcal{I}(A) = I \cap A' \cap A$, and A' is club, so $I \cap A' \in F$, and $A \in F$, so that $I \cap A' \cap A \in F$, and thus $\mathcal{I}(A) \in F$.

For the other direction, if κ is greatly inaccessible, then the uniform, normal filter, F, on κ, contains all the club subsets of κ. Then, since $\kappa \in F$, and $\mathcal{I}(\kappa) = I \in F$, and for any club $D \subseteq \kappa$, the set D is in F, and their intersection, $D \cap I$, is in F, hence $D \cap I$ is non-empty. Thus I is stationary in κ, and thus κ is Mahlo. □

Theorem 3.2 below shows that a Mahlo cardinal has every degree of inaccessibility defined previously. However, a cardinal being every degree of inaccessibility is not equivalent to the cardinal being Mahlo. Theorem 3.3 separates the two notions with forcing to destroy the Mahlo property of a cardinal, while preserving that the cardinal has every inaccessible degree.

Theorem 3.2 *If κ is Mahlo, then for every meta-ordinal t having ordinal terms less than κ, the cardinal κ is a t-inaccessible cardinal (as in Definition* 2.7*).*

Proof. If κ is Mahlo, then κ is greatly inaccessible, by Theorem 3.1. Hence, there is a uniform, normal filter F on κ, closed under the inaccessible limit point operation. Since $\kappa \in F$, and $\mathcal{I}(\kappa) = I \in F$, all sets of α-inaccessible cardinals for $\alpha < \kappa$, below κ, are in F. And, since F is normal, all sets of hyper inaccessible degrees below κ, which are all definable from diagonal intersection, are in F. Hence, every set of inaccessible cardinals definable from $I, \mathcal{I}$, and Δ, in the manner of the previous section, by closing under these operations, is unbounded in κ. □

Theorem 2.1 shows that any cardinal which is at least α-inaccessible in the ground model, for some α, can be killed softly to be no more than α-inaccessible in a forcing extension. Therefore, if κ is α-inaccessible, then there is a forcing extension where κ is α-inaccessible, but not Mahlo. The following theorem shows that if κ is Mahlo, there is a forcing extension where κ is no longer Mahlo but still any degree of inaccessibility defined in the previous section.

Theorem 3.3 *If κ is Mahlo, then there is a forcing extension where κ is t-inaccessible for every meta-ordinal term t with ordinals less than κ , but where κ is not Mahlo.*

P r o o f .   Let κ be a Mahlo cardinal. Let $\mathbb{C}$ be the forcing to add a club, $\mathcal{C} = \bigcup G$, where $G \subseteq \mathbb{C}$ is V-generic, which contains no inaccessible cardinals. The argument in Theorem 2.1 shows that $\mathbb{C}$ preserves cardinals, cofinalities, and all inaccessible cardinals in V_κ. In fact, $\mathbb{C}$ does not change V_κ at all, $V_\kappa^{V[G]} = V_\kappa$. Since V_κ has all the sets needed to define that κ is α-inaccessible, for cardinals $\gamma < \kappa$, being α-inaccessible is absolute to V_κ. Hence $\mathbb{C}$ preserves all inaccessible degrees. Hence κ is still every possible inaccessible degree. However, κ is no longer Mahlo in $V[G]$, since $C \cap I = \varnothing$. □

Just as there are infinitely many degrees of inaccessible cardinals, there are infinitely many Mahlo degrees. One might be tempted to define the next degree of Mahlo to be a Mahlo limit of Mahlo cardinals, exactly as was done with the degrees of inaccessible cardinals, and indeed there is a hierarchy of Mahlo cardinals that can be defined this way. However, there is a more powerful and appropriate way to define the degrees of Mahlo cardinals which is much stronger. The definition of Mahlo cardinal is primarily about stationary sets, and the degrees of Mahlo cardinals are classically defined by stationary sets. Thus, the classical degrees of Mahlo cardinals are defined using stationary sets. Namely, an infinite cardinal κ is *1-Mahlo* if and only if κ is Mahlo, and the set of Mahlo cardinals below κ is stationary in κ. In general, κ is *α-Mahlo* if and only if κ is Mahlo, and for every $\beta < \alpha$, the set of β-Mahlo cardinals below κ is stationary in κ. The degrees of Mahlo cardinals go on forever, just as the degrees of inaccessible cardinals; if κ is κ-Mahlo, then κ is *hyper-Mahlo*, and so on.

The main theorem of this section will show how to force to change degrees of Mahlo cardinals by adding a club avoiding a stationary set of cardinals of a certain Mahlo degree, while preserving all stationary subsets of cardinals of a lesser Mahlo degree. A modification of the forcing $\mathbb{C}$, to add a club, from the proof of Theorem 2.1, will work. We shall need the following lemma.

Lemma 3.4 *Let κ be an inaccessible cardinal and let $S \subseteq \kappa$ be an unbounded subset of κ which contains the singular cardinals. Then $\mathbb{Q}_S$, the forcing to add a club $C \subset S$, preserves cardinals and cofinalities, and for all $\beta < \kappa$, the forcing $\mathbb{Q}_S$ has a $\leq \beta$-closed dense subset.*

P r o o f .   Conditions $c \in \mathbb{Q}_S$ are closed, bounded subsets of S. The ordering on $\mathbb{Q}_S$ is end-extension: $d \leq c$ if and only if $c = d \cap (\sup(c) + 1)$. Let $G \subseteq \mathbb{Q}_S$ be V-generic, and let $C = \bigcup G$. Then $C \subseteq S$ is club in κ. The proof is the same as in Theorem 2.1. □

Given two sets A and B, it is said that A *reflects* B if for some $\delta \in B$, the set $A \cap \delta$ is stationary in δ. The following Lemma shows that if A does not reflect in B, then the forcing to add a club avoiding B will preserve the stationary subsets of B.

That is, if we force to add a club which avoids the strong set, then the stationarity of the weak set and all of its stationary subsets will be preserved if the strong set does not reflect in the weak set. In the proof of the theorem, the strong set will be Mahlo cardinals of a fixed degree, and the weak set will be the set of Mahlo cardinals of a lesser degree which are not of the fixed degree.

Lemma 3.5 *If κ is Mahlo, and the sets A and B partition the inaccessible cardinals below κ, where A does not reflect in B, then the forcing to add a club avoiding A will preserve all the stationary subsets of B. Furthermore, the forcing does not add sets to V_κ.*

P r o o f .   Suppose $S \in V$ is a stationary subset of B. Let $\mathbb{P}$ be the forcing to add a club through the complement of A, and let $\dot{E}$ be a name for a club subset of κ, and p a condition which forces that E is club in κ. Let $\delta \in S$ be such that $p \in V_\delta$ and $\langle V_\delta, \in, A \cap \delta, B \cap \delta, \dot{E} \cap V_\delta \rangle$ is an elementary substructure of $\langle V_\kappa, \in, A, B, \dot{E} \rangle$. We can find such $\delta \in S$, since the set of δ giving rise to elementary substructures is club and S is stationary. Since $\delta \in B$, by the assumption, there is a club set $c \subseteq \delta$, containing no points from A. Now we construct a pseudo-generic δ-sequence of conditions below δ, deciding more and more about $\dot{E}$, using the elements of c to guide the construction. Build a descending sequence of conditions $\langle c_\alpha \ : \ \alpha < \delta \rangle$ in $\mathbb{P} \cap V_\delta$, below p, and given c_α, choose $c_{\alpha+1}$ to force a specific ordinal above α into $\dot{E}$ with $\sup(c_{\alpha+1}) \in c$, and at limits λ, let $c_\lambda = (\bigcup_{\alpha<\lambda} c_\alpha) \cup \{\sup(\bigcup_{\alpha<\lambda} c_\alpha\}$. Notice that $\sup(\bigcup_{\alpha<\lambda} c_\alpha) \in c$, since c is closed, and therefore is not in A, so that c_λ is a condition for limit ordinals λ, in the construction. That is, we can get through the limit steps below δ, precisely because c contains no points from A. Let $c^* = (\bigcup_{\alpha<\delta} c_\alpha) \cup \{\delta\}$. Then $c^* \subseteq \delta$ is club in δ that decides $\dot{E}$ in a way that is unbounded in δ. Thus, c^* is a condition which forces $\dot{E}$ meets S. Thus, S must still be stationary in the extension. Finally, this club shooting forcing does not add sets to V_κ by Lemma 3.4. □

The following is the main killing-them-softly result of this section; this result had been proved earlier by Boos [1]:

Theorem 3.6 *If κ is α-Mahlo, then there is a forcing extension where κ is α-Mahlo, but not $(\alpha + 1)$-Mahlo.*

Proof. Let $\alpha < \kappa$ be fixed. Suppose κ is α-Mahlo. Let A be the set of α-Mahlo cardinals below κ. Note that A does not reflect in its complement in the inaccessible cardinals, for if, for some $\beta < \kappa$, inaccessible, in the complement of A, the set of α-Mahlo cardinals below β is stationary, then β is $(\alpha + 1)$-Mahlo, hence α-Mahlo, hence $\beta \in A$. But, β was supposed to be an inaccessible cardinal in the complement of A. Thus, A does not reflect in the set of inaccessible cardinals in its complement.

If κ is not $(\alpha + 1)$-Mahlo, then force trivially to show the result. Let $\mathbb{C}$ be the notion of forcing which adds a club $\mathcal{C}$ through the complement of A. Conditions are ordered by end-extension. Then by Lemma 3.4 the forcing $\mathbb{C}$ preserves truth in V_κ, preserves cardinals, cofinalities, and adds a club to κ, disjoint from A. Since A is no longer stationary in κ, it is no longer $(\alpha + 1)$-Mahlo. Since A contains no clubs, $\kappa \setminus A$ is stationary. Thus, by Lemma 3.5, since A does not reflect in its complement, all stationary subsets of $\kappa \setminus A$ are preserved. Thus, I only need to show that $\forall \beta < \alpha$ the set $T_\beta = \{ \delta < \kappa \ : \ \delta \text{ is } \beta\text{-Mahlo but not } (\beta + 1)\text{-Mahlo} \}$ is stationary. Let $C \subseteq \kappa$ be club. Let δ be least in C' which is β-Mahlo. Then $C \cap \delta$ is club in δ so that $(C \cap \delta)' \cap \delta = C' \cap \delta$ is club in δ and contains no β-Mahlo cardinals by the minimality of δ. Thus δ is not $(\beta + 1)$-Mahlo. Thus T_β is stationary. Thus the stationarity of T_β is preserved. Thus κ is still α-Mahlo. □

As in the previous section for inaccessible cardinals (Definition 2.5), we can define the higher degrees of Mahlo cardinals:

Definition 3.7 A cardinal κ is α-hyper$^\beta$-*Mahlo* if and only if

(1) the cardinal κ is Mahlo, and
(2) for all $\eta < \beta$, the cardinal κ is κ-hyper$^\eta$-Mahlo, and
(3) for all $\gamma < \alpha$, the set of γ-hyper$^\beta$-Mahlo cardinals below κ is stationary in κ.

Also, as in Definition 2.7 for degrees of inaccessible cardinals, we can define the classes of degrees of Mahlo cardinals uniformly by using meta-ordinals:

Definition 3.8 If t is a meta-ordinal term having only ordinals less than κ, then a cardinal κ is t-Mahlo if and only if for every meta-ordinal term $s < t$ having only ordinals less than κ, the set of s-inaccessible cardinals is stationary in κ.

Theorem 3.9 *If κ is t-Mahlo, where t is a meta-ordinal term having parameters less than κ, then there is a forcing extension $V[G]$ where κ is t-Mahlo, but not $(t + 1)$-Mahlo.*

Proof Sketch. The proof is a generalization of the proof of Theorem 3.6. The proof is to add a club $C \subseteq \kappa$ which avoids the t-Mahlo cardinals below κ. □

Finally, we can go beyond to define the higher degrees of Mahlo cardinals, just as we did with the higher degrees of inaccessible cardinals, using the new words and meta-ordinals:

κ is Ω-Mahlo if and only if κ is hyper-Mahlo,
κ is Ω^2-Mahlo if and only if κ is richly-Mahlo,
κ is Ω^3-Mahlo if and only if κ is utterly-Mahlo,
κ is Ω^4-Mahlo if and only if κ is deeply-Mahlo,
κ is Ω^5-Mahlo if and only if κ is truly-Mahlo,
κ is Ω^6-Mahlo if and only if κ is eternally-Mahlo, and
κ is Ω^7-Mahlo if and only if κ is vastly-Mahlo.

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