Complex Crossings

1 Introduction

The graph of a complex function requires four perpendicular axis if we are to see it in the way we can fully see any real function on the plane. In this paper we will see the graphs of complex functions in a compromised space, i.e. in three dimensions, and explore the self-intersections of these graphs. One way to do this is to have two of the required four axes share one axis in space. Another way is to use the three perpendicular axes of space and consider the fourth to be in the *normal* direction which will be relative to each domain value.

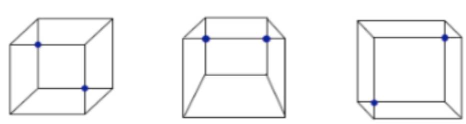
First, a discussion of self-intersection or crossings in a graph. The graph of a real function of one variable does not cross itself in the plane. Similarly, a real function of two variables does not self-interesect in space. In the same way, a complex function of two variables does not intersect itself in four dimensional "space". Consider a complex function written parametrically:

$$f(x, y) = (x, y, u(x, y), v(x, y))$$

where u and v are the real and imaginary parts of f, respectively. If the graph of f were to intersect itself we would have to have $x_1 \neq x_2$ and/or $y_1 \neq y_2$ such that

$$(x_1, y_1, u(x_1, y_2), v(x_1, y_1)) = (x_2, y_2, u(x_2, y_2), v(x_2, y_2)).$$

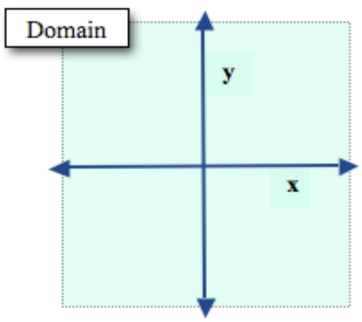
So, the graph of a complex function does not cross itself in four dimensional "space", in it's natural home. However, when we use either of the methods mentioned in the first paragraph we get self-intersection. The crossings in the three dimensional representations are the consequences of not enough room, but provide perspective in the same way that a drawing of a cube in two dimensions would have intersecting lines.

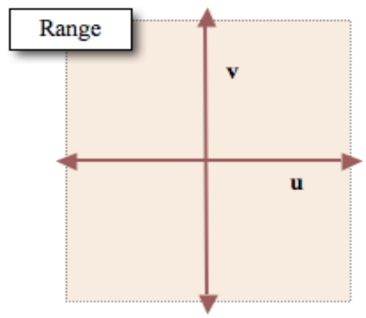


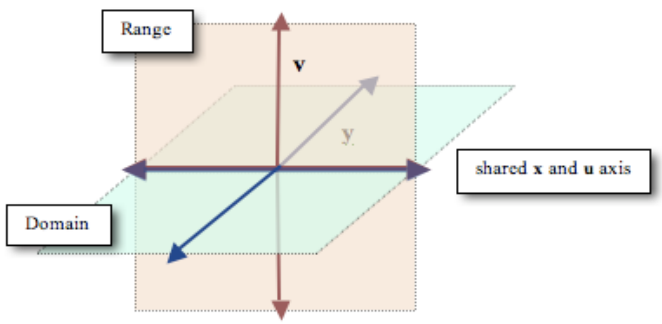
Notice that where the lines cross in the drawing of the cube depend on the perspective. We should also be aware that there are three dimensional representations of complex functions that do not self-intersect, such as the graphs of the real or imaginary parts of the function, which in this way demonstrates an important quality of the function, but in other ways does not display its four dimensional essence. In the same way that a square is a two dimensional representation of a cube, and does not have intersecting lines. But a square does not point to the spaciousness of the cube. In fact, the places where the two dimensional representation of the cube (above) has crossings points to spaciousness of the cube, and highlights the difference of the dimensions: a crossing in two dimensions only means behind in three dimensions.

2 Shared Axis Representation

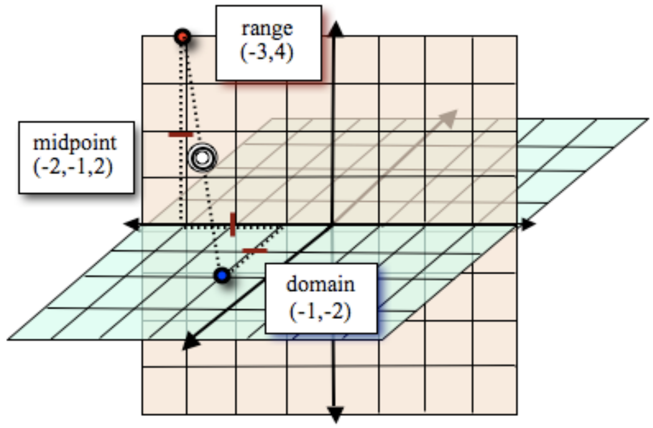
First, let us consider the three dimensional representation of the graph of a complex function by a *shared axis representation* and then we will see where it has self-intersection. Consider the complex plane representing the domain of a complex function, with a real x-axis and an imaginary y-axis. Suppose a complex function f is written in terms of its real and imaginary parts: $f(z) = u + iv$. Then the complex plane representing the range of the complex function has a real u-axis and an imaginary v-axis. Now, put these planes together. Put the domain plane through the range plane so that the real axis of the domain plane is the same as the real axis of the range plane (negative and positive parts of the axes match). Now we have a three dimensional space that involves all four axes.





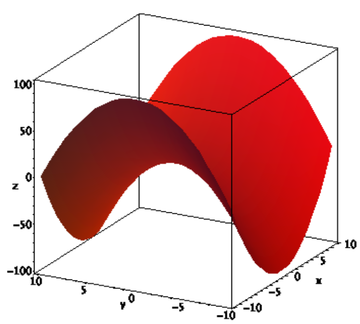


Then, to see how to use this space to create a representation of the complex graph follow this example: Consider the function $f(z) = z^2$ and domain value $-1-2i$. We have $f(-1-2i) = (-1-2i)^2 = -3+4i$. Plot these two points on the intersecting planes described above ($-1-2i$ on the domain plane and $-3+4i$ on the range plane) and plot he midpoint of the line connecting them. Plot all the midpoints between the domain points and their corresponding range points to get a three dimensional graph of the function. Midpoint of example is shown below.

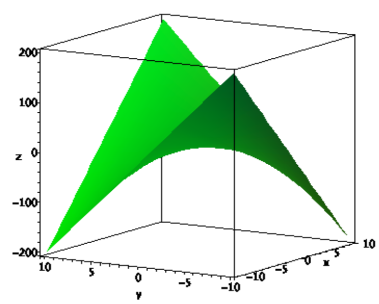


Let $x+iy$ or (x,y) be any point in the domain of the function z^2. The real and imaginary parts of the function $f(z)=z^2$ are given by $f(x,y)=u+iv$ where $u=x^2-y^2$ and $v=2xy$. Then the real part of the the graph of z^2 is given parametrically by $u(x,y)=(x,y,x^2-y^2)$, and the imaginary part is given by $v(x,y)=(x,y,2xy)$.

Real Part $f(z)=z^2$



Imaginary Part $f(z)=z^2$

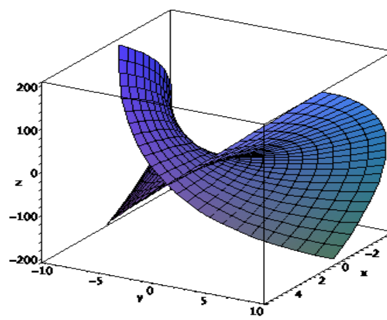


The shared axis representation of any complex function $f(z)=u+iv$ which is the graph of all the midpoints described above is given by the general rule:

$$s(x,y)=(1/2(x+u), 1/2y, 1/2v)$$

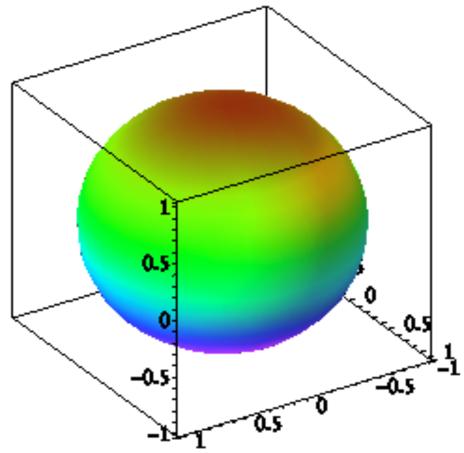
The first coordinate is the shared real parts, the second is the domain imaginary part, and the third is the range imaginary part. The shared axis representation of $f(z) = z^2$ is $s(x, y) = (1/2(x + x^2 - y^2), 1/2y, xy)$ and is depicted below.

Representation Shared Real Axis $f(z) = z^2$

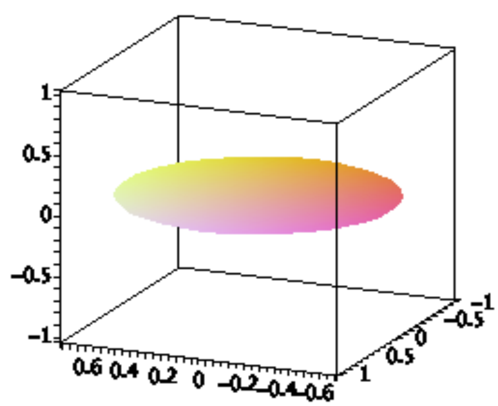


3 Analogy to Three Dimensions

The method described above can be applied to a three dimensional object. In doing, so one could see a representation of the three dimensional object in just two dimensions. Three dimensional objects are often represented in two dimensions using shading or coloring. For example, with the appropriate shading a sphere can be represented with a circle. In this example the sphere is described with three axes x, y and z. In the case of a complex function, we combined two of the four axes to get a 3-D representation. If we use the method above to represent the sphere in two dimensions, two of the axes must be combined into one. The shere is given parametrically by $(\cos x \sin x, \sin x \sin y, \cos y)$ and depicted below (of course this is already a 2-D representation of the sphere).



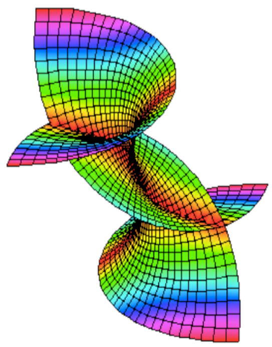
The representation of the sphere in the plane using the above method combines the x and z axes. The two dimensional representation which combines the x and z axis is given parametrically by $\left(\frac{1}{2}(\cos x \sin y + \cos y), 1/2(\sin x \sin y)\right)$ and represented below.



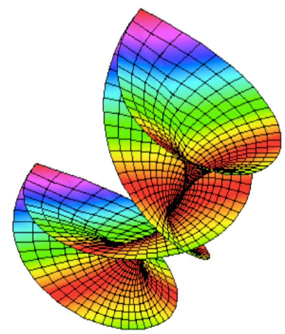
4 Shared Axis Representation Crossings

Here are some more examples of graphs of complex functions created by the shared axis method described in the previous section.

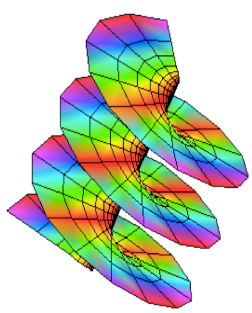
$f(z) = z^3$



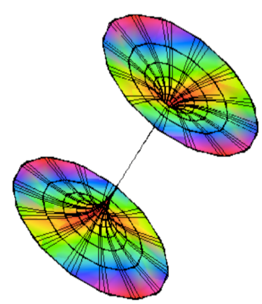
$f(z) = z^4$



$f(z) = e^z$



$f(z) = \sin z$



All of these, and the example $f(z) = z^2$ from the last section have self-intersections. Using the cube analogy, these crossings point to the places where there is some "four dimensional space" in the true graph of the function that cannot be described in three dimensions. These graphs look very much like the graphs of the *imaginary part of the inverse* of the these complex polynomial functions, i.e. graphs with parametric equations $r(x, y) = (u(x, y), v(x, y), y)$. This happens because when x and y are allowed to range over a large interval, the prominent term in the first coordinate $x + u(x, y)$ is determined by $u(x, y)$. More on the range of x and y and how it affects the shape of the shared axis graphs later. For now we are interested in where these graphs have crossings.

Consider a complex function f written as $f(x, y) = (u(x, y), v(x, y))$. The shared axis graph is given by

$$s(x, y) = (1/2(x + u(x, y)), 1/2y, 1/2v(x, y))$$

The crossings occur when, for some $x_1 \neq x_2$ or $y_1 \neq y_2$ we have

$$x_1 + u(x_1, y_1) = x_2 + u(x_2, y_2)$$

$$y_1 = y_2$$

$$v(x_1, y_1) = v(x_2, y_2)$$

Since the second equation above cannot hold with the assumption $y_1 \neq y_2$, let y be some constant, say $y = t$. Also, let us assume that $t \neq 0$. Then we only need to solve these two equations for x_1 and x_2 to find the crossings:

$$x_1 + u(x_1, t) = x_2 + u(x_2, t)$$
$$v(x_1, t) = v(x_2, t)$$

For example, let $f(z) = z^3 = (x + iy)^3$. the the real and imaginary parts are

$$u(x, y) = x^3 - 3xy^2$$

$$v(x, y) = 3x^2y - y^3$$

The parametric equation of the shared axis graph (depicted at the beginning of the section) is

$$s(x, y) = (1/2(x + x^3 - 3xy^2), 1/2y, 1/2(3x^2y - y^3))$$

Then to find where this graph has self-intersection solve these equations for x_1, x_2 where $x_1 \neq x_2$.

$$x_1 + x_1^3 - 3x_1t^2 = x_2 + x_2^3 - 3x_2t^2 \tag{1}$$

$$3x_1^2t - t^3 = 3x_2^2t - t^3 \tag{2}$$

From (2) we have $x_2^2 = x_1^2$ which implies $x_2 = \pm x_1$ which implies $x_2 = -x_1$. Putting this into (1) and solving for x_1 gives

$$x_1(1 + x_1^2 - 3t^2) = 0$$

Which means either $x_1 = 0$ or $x_1 = \pm\sqrt{3t^2 - 1}$. Since we do not want $x_1 = x_2$ we exclude the case $x_1 = 0$ and are left with these solutions:

$$x_1 = \pm\sqrt{3t^2 - 1} \qquad x_2 = \mp\sqrt{3t^2 - 1}$$

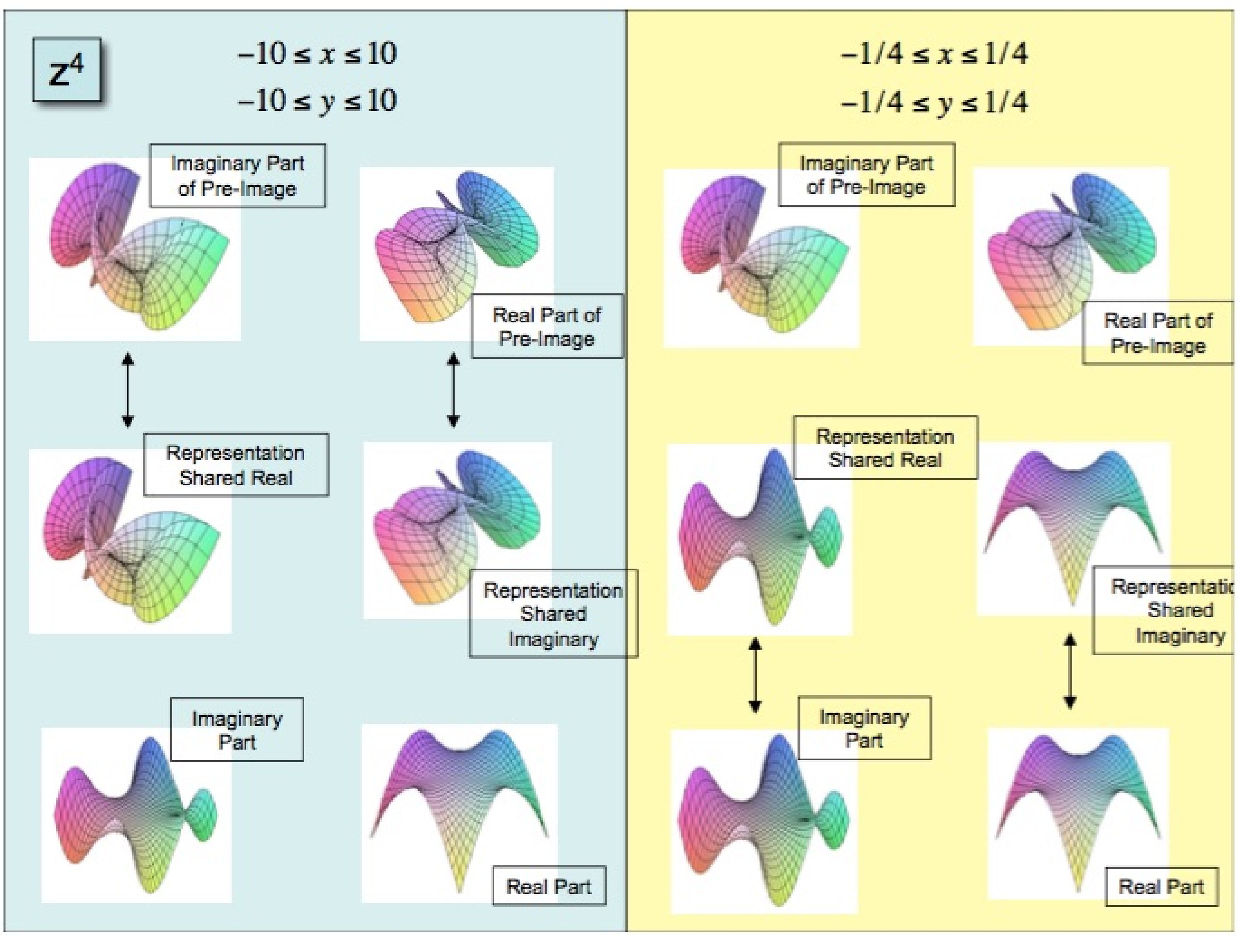
Then, the shared axis graph has self-intersection at the following space curve

$$\begin{aligned} c(t) &= (1/2(\sqrt{3t^2 - 1} + (\sqrt{3t^2 - 1})^3 - 3t^2\sqrt{3t^2 - 1}), t, 3(3t^2 - 1)t - t^3) \\ &= (0, t, 8t^3 - 3t) \end{aligned}$$

Notice that the domain restriction on this curve is $3t^2 - 1 \geq 0$ which implies $t \geq \frac{\sqrt{3}}{3}$ or $t \leq -\frac{\sqrt{3}}{3}$. Since $y = t$ this is a restriction on where the crossings happen in terms of the values of y. In fact, when we look at any of these shared axis graphs for complex polynomials for values of x and y in a small window about zero we will not see any crossings. That is, if we "zoom in" parametrically (or go back in space and time if we think of x and y as these parameters) we see a manifold that looks like the graph of the imaginary part of the function given by $p(x, y) = (x, y, v(x, y))$.

The reasoning is the same as the previous discussion as to why the shared axis graph looks like the imaginary part of the inverse when we give x and y a bigger range: the leading term of the first coordinate depends on the values of x and y. So, when x and y are smaller than one and get closer to zero, the term with the most pull in the first coordinate $x + u(x, y)$ will be x if the degree of the power series expansion of $u(x, y)$ is greater than one.

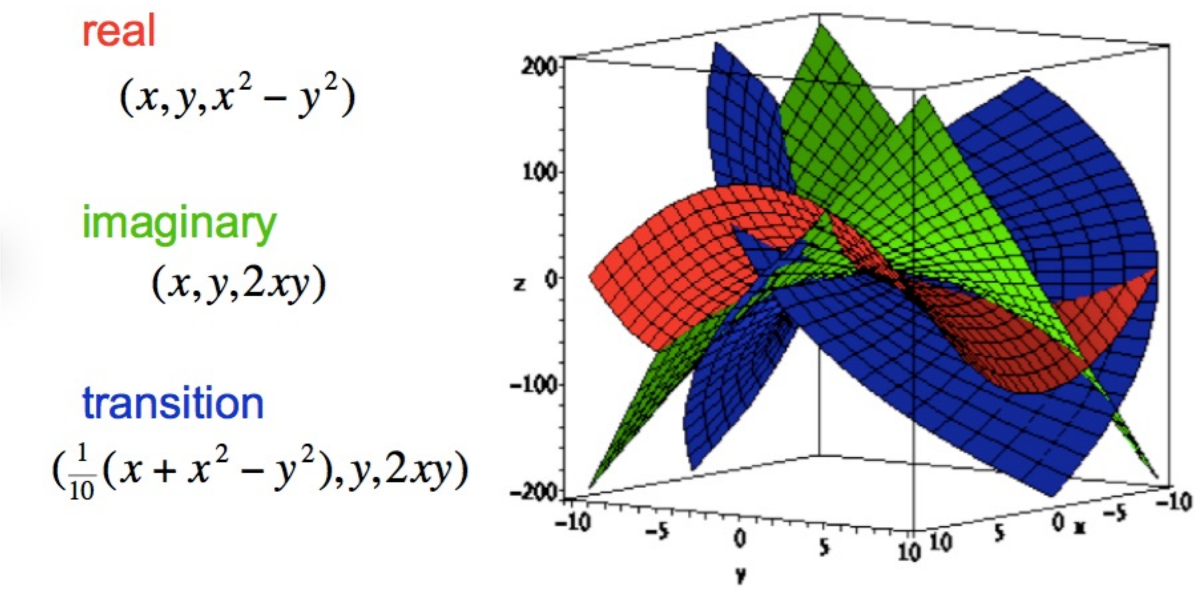
The following diagram shows the effects of a smaller window for the shared axis representation for $f(z) = z^4$



5 Mathematics of Animation

The graphs produced above with the described method can also be thought of as transition graphs between the real part and the imaginary part of the graph of a complex function. Consider a complex function $f(z) = u + iv$ and a point (x, y) in its domain. The graph of the real part of this function is given by (x, y, u) and the graph of the imaginary part is given by (x, y, v). The graph of the transition is a mixture of these two and is again $(1/2(x + u), 1/2(y), 1/2(v))$. Since the transition graph has the same basic shape no matter what the constant in front of the variables is, we may think in general with the transition graph given by $(a(x + u), by, cv)$. The constants a, b, and c will be determined and may change (for scaling purposes) depending on the complex function.

Animations were created to show the change from the real graph to the imaginary graph with the transition halfway through. First, to get the correct scaling, the constants a, b, c were determined for each particular function. To do this, all three graphs (real, imaginary, transition) were plotted on the same axes. For example, below are the three graphs for $f(z) = z^2$ with the transition given by $(1/20(x + x^2 - y^2), y, 2 * x * y)$.



Once the correct constants a, b, and c were found, an annimation was created that began with the real part of the function and ended with the imaginary with the transition inbetween. Consider the complex function in parametric form (x, y, u, v) and as the real, imaginary and transition graphs as linear transformations of the complex function from four dimensions to three. Then the matrix for the real part is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The matrix for the transition is

$$\begin{bmatrix} a & 0 & a & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & c \end{bmatrix}$$

And the matrix for the imaginary part is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then to annimate between these we need a matrix with variables that change from the real to the transition to the imaginary. This variable matrix is given by

$$\begin{bmatrix} M & 0 & N & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & Q & R \end{bmatrix}$$

where M, N, P, Q and R will change with time. If the time t of annimation is from $t = 0$ to $t = 2$ then we would like to have the real matrix when $t = 0$, the transition matrix when $t = 1$ and the imaginary matrix when $t = 2$. Then thinking of M, N, P, Q and R as functions of time we have the following

$$M(0) = 1; M(1) = a; M(2) = 1$$

$$N(0) = 0; N(1) = a; N(2) = 0$$

$$P(0) = 1; P(1) = b; P(2) = 1$$

$$Q(0) = 1; Q(1) = 0; Q(2) = 0$$

$$R(0) = 0; R(1) = c; R(2) = 1$$

We can use linear interpolation to find polynomials that satisfy the above conditions. The lagrange interpolating polynomials for the interpolation points $t = 0, 1, 2$ are given by

$$L_1(t) = \frac{1}{2}(t-1)(t-2)$$

$$L_2(t) = -t(t-2)$$

$$L_3(t) = \frac{1}{2}t(t-1)$$

Thus the interpolating polynomials are given by

$$\begin{aligned} m(t) &= M(0)L_1(t) + M(1)L_2(t) + M(2)L_3(t) \\ &= \frac{1}{2}(t-1)(t-2) - at(t-2) + \frac{1}{2}t(t-1) \\ &= (1-a)t^2 - (2-2a)t + 1 \end{aligned}$$

$$
\begin{aligned}
n(t) &= N(0)L_1(t) + N(1)L_2(t) + N(2)L_3(t) \\
&= -at(t-2)
\end{aligned}
$$

$$
\begin{aligned}
p(t) &= P(0)L_1(t) + P(1)L_2(t) + P(2)L_3(t) \\
&= \frac{1}{2}(t-1)(t-2) - bt(t-2) + \frac{1}{2}t(t-1) \\
&= (1-b)t^2 - (2-2b)t + 1
\end{aligned}
$$

$$
\begin{aligned}
q(t) &= Q(0)L_1(t) + Q(1)L_2(t) + Q(2)L_3(t) \\
&= \frac{1}{2}(t-1)(t-2) \\
&= \frac{1}{2}t^2 - \frac{3}{2}t + 1
\end{aligned}
$$

$$
\begin{aligned}
r(t) &= R(0)L_1(t) + R(1)L_2(t) + R(2)L_3(t) \\
&= -ct(t-2) + \frac{1}{2}t(t-1) \\
&= (\frac{1}{2} - c)t^2 + (2c - \frac{1}{2})t
\end{aligned}
$$

Then the variable matrix is given by

$$
\begin{bmatrix}
m(t) & 0 & n(t) & 0 \\
0 & p(t) & 0 & 0 \\
0 & 0 & q(t) & r(t)
\end{bmatrix}
$$

and the parametric form of the varible function is $(m(t)x+n(t)u, p(t)y, q(t)u+r(t)v)$. For example for the complex function z^2 it is

$$(m(t)x + n(t)(x^2-y^2), p(t)y, q(t)(x^2-y^2) + r(t)(2xy))$$

$$= ([(1-1/20)t^2-(2-2(1/20))t+1]x-1/20t(t-2)(x^2-y^2), y, [\frac{1}{2}t^2-\frac{3}{2}t+1](x^2-y^2)+[-t^2+3t](xy)).$$

The Maple command for this annimation is

$$animate3d([[(1-1/20)*t^2-(2-2*(1/20))*t+1]*x-1/20*t*(t-2)*(x^2-y^2), y, [1/2*t^2-3/2*t+1]*(x^2-y^2)$$

$$+[-t^2+3*t]*(x*y)], x=-10..10, y=-10..10, t=0..2);$$

6 References

T. Banchoff, *Beyond the Third Dimension: Geometry, Computer Graphics, and Higher Dimensions,* Scientific American Library, 1990.

J. Mathews and R. Howell, *Complex Analysis for Mathematics and Engineering* (5th ed.), Jones and Bartlett Publishers, Inc., 2006.

W. Rudin, *Real and Complex Analysis* (3rd ed.), WCB/McGraw-Hill, 1987.