Set Theory: from Cantor to Cohen and beyond

Math Club
Fordham University
April 30, 2019

Erin Carmody

Outline

1. Transcendental numbers exist. (Cantor)
2. Cantor's Theroem.
3. WO $\equiv$ AC. (Zermelo)
4. Faith in Large Cardinals.
5. The Independence of CH. (Cohen)
6. Degrees of inaccessible cardinals.
7. Killing Them Softly Theorems. (Carmody)

First Theorem (Cantor)

The real numbers are uncountable
(proof of the existence of transcendental numbers).

First Theorem (Cantor)

Proof: We will show that the set $(0, 1)$ is uncountable (and hence so is $\mathbb{R}$, since they have the same size).

First Theorem (Cantor)

Suppose, to the contrary, that $(0,1)$ is countable. Then we can write these numbers on a countable list:

$$\{s_1, s_2, s_3, \dots\}$$

First Theorem (Cantor)

We will find a number not on the list by diagonalizing:

$$t = 0.d_1 d_2 d_3 \ldots d_n \ldots$$

where

$$d_1 \text{ is different from } s_{11}$$

$$d_2 \text{ is different from } s_{22}$$

$$\vdots$$

$$d_n \text{ is different from } s_{nn}$$

$$\vdots$$

First Theorem (Cantor)

Then the nth digit of t is different from the nth digit of s_n.

Thus, $t \neq s_k$ for any k.

Thus $t \in (0, 1)$, but t is not on the list.

Contradiction.

First Theorem (Cantor)

Thus, the reals are uncountable.

(And since the algebraic numbers are countable,
there must be
uncountably many transcendental numbers,
hence they exist.)

Second Theorem (Cantor)

Theorem: Every set X has cardinality less than its power set $P(X)$.

Proof: Let f be a function from X into $P(X)$

$$f : X \to P(X).$$

The function cannot be a one-to-one map onto $P(X)$. For, consider the set $Y = \{x \in X \mid x \notin f(x)\}$.

Suppose Y is in the range of f. Then there is some z for which $f(z) = Y$.

Then $z \in Y \iff z \notin Y$, a contradiction.

Ordinals

$$\omega = \{1, 2, 3, \dots\}$$

$$\cdots\cdots >$$

$$\omega + 1 = \{1, 2, 3, \dots \omega\}$$

$$\cdots\cdots > \cdot$$

$$\omega + 2 = \{1, 2, 3, \dots \omega, \omega + 1\}$$

$$\cdots\cdots > \cdot\ \cdot$$

Ordinals

$$\omega \cdot 2 = \{1, 2, 3, \ldots \omega, \omega + 1, \omega + 2, \omega + 3, \ldots\}$$

$$\cdots\cdots > \cdots\cdots >$$

$$\omega \cdot 3 = \{1, 2, 3, \ldots \omega, \omega + 1, \omega + 2, \ldots \omega \cdot 2, \omega \cdot 2 + 1, \ldots\}$$

$$\cdots\cdots > \cdots\cdots > \cdots\cdots >$$

Ordinals

$$\omega \cdot \omega = \{1, 2, 3, \ldots \omega, \omega + 1, \omega + 2, \ldots \omega \cdot 2, \ldots \omega \cdot 3, \ldots \omega \cdot n, \ldots\}$$

$$\cdots\cdots > \cdots\cdots > \cdots\cdots > \;\; ... \;\; \cdots\cdots > ...$$

Third Theorem (Zermelo)

The Axiom of Choice
is equivalent to the
Well-Ordering Principle.

Third Theorem (Zermelo)

($\Rightarrow$) Let A be a nonempty set. We would like to enumerate A.
Pick any element of A to be the first element a_0.

If $a_0, a_1, \ldots, a_\alpha$ have been chosen,
choose $a_\alpha \in A - \{a_0, a_1, \ldots, a_\alpha\}$.

If λ is a limit ordinal and $\{a_\beta \mid \beta < \lambda\}$ has been made from A,
choose $a_\lambda \in A - \{a_\beta \mid \beta < \lambda\}$.

Let θ be the least ordinal such that $A = \langle a_\alpha \, | \alpha < \theta\rangle$.
Thus, A has been enumerated by θ.

Third Theorem (Zermelo)

($\Leftarrow$) Suppose that every set can be well-ordered.
Let S be a family of non-empty sets.

Let $\cup S$ be the set of elements of each of the families in S.
Well order $\cup S$ and thus every family in S has been well ordered.

Let $f(X)$ be the least element of X for every $X \in S$.
Thus, f is a choice function for the family S.

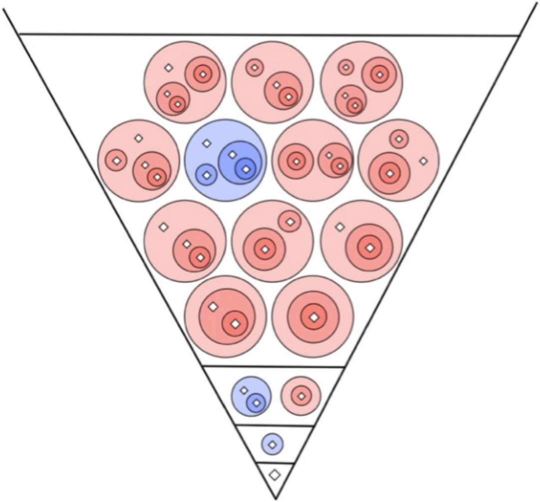

Figure: Hierarchy of Sets

$V_\kappa \models ZFC$

An *inaccessible* cardinal is an
uncountable,
regular,
strong limit
cardinal.

An uncountable version of ω.

$V_\kappa \models \mathsf{ZFC}$

V_ω almost satisfies the axioms of ZFC.

Except for the axiom of infinity.

$V_\kappa \models$ ZFC

Theorem: If κ is inaccessible $V_\kappa \models$ ZFC.

For every limit ordinal α, the set V_α satisfies: empty set, extensionality, pairing, union, foundation, and choice.

If $\alpha > \omega$, then $V_\alpha \models \infty$.

Since κ is a strong limit, V_κ satisfies the power set axiom.

The most difficult part: showing V_κ satisfies the replacement axiom.

We cannot prove inaccessible cardinals exist.

Corollary: We cannot prove the existence of inaccessible cardinals, if ZFC is consistent.

Proof: Suppose we could prove that an inaccessible cardinal κ exists. Then, by the theorem, $V_\kappa \models$ZFC. So, we have proved the existence of a set model of ZFC. Thus, we would have proved, using the axioms of ZFC, that ZFC is consistent. This contradicts Gödel's second incompleteness theorem. Thus, we cannot prove that inaccessible cardinals exist.

Forcing Conditions

Let $(\mathbb{P}, <)$ be a nonempty partially ordered set in M (the *ground model*, a transitive model of ZFC)

$(\mathbb{P}, <)$ is a *notion of forcing*
and the elements of $\mathbb{P}$ are *forcing conditions*.

For any $p, q \in \mathbb{P}$, we say p is *stronger* than q if $p < q$.

If there exists an $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$,
then p and q are *compatible*.

Dense Sets

A set $D \subset \mathbb{P}$ is *dense* in $\mathbb{P}$ if for every $p \in \mathbb{P}$
there is a condition $q \in D$ which is stronger than p.

Filters

A set $F \subset \mathbb{P}$ is a *filter* on $\mathbb{P}$ if

F is nonempty;
if $q \leq p$ and $p \in F$, then $q \in F$;
any two elements of F are compatible.

Generic Sets

A set $G \subset \mathbb{P}$ is *generic* over M if G is a filter, and if $D \in$ M is dense in $\mathbb{P}$, then $G \cap D \neq \emptyset$.

Adding a Cohen Real

(Proof Sketch)
Let $\mathbb{P}$ be defined as follows:

$$p \in \mathbb{P} \iff p \text{ is a finite binary sequence.}$$

For $p, q \in \mathbb{P}$, the condition p is stronger than q if p extends q:

$$p < q \iff p = q^\frown s$$

(p and q are compatible if $p < q$ or $q < p$)

Adding a Cohen Real

Suppose $(\mathbb{P}, <) \in \mathrm{M}$, the ground model.

Let $G \subset \mathbb{P}$ be generic over M.

Let $f = \cup G$.

Since G is a filter, f is a function because the conditions in a filter are compatible.

Adding a Cohen Real

For every $n \in \omega$, the following set are dense in $\mathbb{P}$:

$$D_n = \{p \in \mathbb{P} \; : \; n \in dom(p)\}$$

Hence, every D_n meets G, so the domain of f is ω.

Adding a Cohen Real

We claim that the function $f : \omega \to \{0,1\}$ is not in M.

For every 0-1 function g in M., let

$$D_g = \{p \in \mathbb{P} \mid p \not\subset g\}$$

Each of these sets is dense in $\mathbb{P}$, and hence meets G.

It follows that $f \neq g$ for any $g \in$ M.

Adding a Cohen Real

We just added a Cohen generic real.

BIG BLACK BOX:

$$M[G] \models ZFC$$

Independence of CH and more

Theorem (Cohen): If $V \models ZFC$, then there is a forcing extension $V[G]$ that satisfies $|\mathbb{R}| > \aleph_1$.

Theorem (Easton): If $V \models GCH$, then there is a forcing extension where the continuum function f can be almost anything as long as it satisfies the following for regular κ and λ:

$$f(\kappa) > \kappa,$$
$$f(\kappa) \leq f(\lambda) \text{ for } \kappa \leq \lambda, \text{ and}$$
$$\mathrm{cf} f(\kappa) > \kappa.$$

Killing Them Softly

Suppose $\kappa \in V$ is a cardinal with large cardinal property A. The main idea is to find a notion of forcing $\mathbb{P}$ such that if $G \subseteq \mathbb{P}$ is V-generic, the cardinal κ no longer has property A in $V[G]$, but has as many of its other large cardinal properties as possible. The main theorems show how to do this for a class of cardinals.

Degrees of Inaccessible Cardinals

κ is *1-inaccessible* if and only if κ is an inaccessible cardinal and a limit of inaccessible cardinals

κ is α-*inaccessible* if and only if κ is inaccessible and for every $\beta < \alpha$, the cardinal κ is a limit of β-inaccessible cardinals

Killing Inaccessible Cardinals Softly

Theorem: If κ is α-inaccessible then there is a forcing extension where κ is still α-inaccessible, but not $(\alpha + 1)$-inaccessible.

Hyper-inaccessible Cardinals

A cardinal κ is *hyper-inaccessible* if and only if κ is κ-inaccessible.

Hyper-inaccessible Cardinals

κ is *1-hyper-inaccessible* if and only if κ is inaccessible and a limit of hyper-inaccessible cardinals

κ is α*-hyper-inaccessible* if and only if κ is inaccessible and $\forall\beta < \alpha$ is a limit of β-hyper-inaccessible cardinals

κ is *hyper*^{2}*-inaccessible* if and only if κ is κ-hyper-inaccessible

Meta-Ordinal Notation

If κ is hyper-inaccessible denote κ as Ω-inaccessible.

If κ is 1-hyper-inaccessible denote κ as $(\Omega + 1)$-inaccessible.

$$\vdots$$

If κ is α-hyper-inaccessible denote κ as $(\Omega + \alpha)$-inaccessible.

$$\vdots$$

Meta-Ordinal Notation

If κ is κ-hyper-inaccessible denote κ as $\Omega \cdot 2$-inaccessible.

$\vdots$

If κ is hyper^3-inaccessible denote κ as $\Omega \cdot 3$-inaccessible.

$\vdots$

If κ is hyper$^\kappa$-inaccessible denote κ as Ω^2-inaccessible.

$\vdots$

Degrees of Inaccessible Cardinals... in words

κ is *richly-inaccessible* $\iff$ κ is hyper$^{\kappa}$-inaccessible

κ is *utterly-inaccessible* $\iff$ κ is richly$^{\kappa}$-inaccessible

κ is *deeply-inaccessible* $\iff$ κ is utterly$^{\kappa}$-inaccessible

κ is *truly-inaccessible* $\iff$ κ is deeply$^{\kappa}$-inaccessible

κ is *eternally-inaccessible* $\iff$ κ is truly$^{\kappa}$-inaccessible

κ is *vastly-inaccessible* $\iff$ κ is eternally$^{\kappa}$-inaccessible

Meta-Ordinal Notation

κ is Ω-inaccessible $\iff$ κ is hyper-inaccessible

κ is Ω^2-inaccessible $\iff$ κ is richly-inaccessible

κ is Ω^3-inaccessible $\iff$ κ is utterly-inaccessible

κ is Ω^4-inaccessible $\iff$ κ is deeply-inaccessible

κ is Ω^5-inaccessible $\iff$ κ is truly-inaccessible

κ is Ω^6-inaccessible $\iff$ κ is eternally-inaccessible

κ is Ω^7-inaccessible $\iff$ κ is vastly-inaccessible

Meta-Ordinal Notation

κ is $(\Omega^7 + \Omega^4 \cdot 3 + 1)$-inaccessible

$\iff$

κ is 1-deeply·3-vastly-inaccessible

References

Apter/Gitik, *The least measurable can be strongly compact and indestructible.* The Journal of Symbolic Logic, Vol. 63, No. 4, 1404-1412, Dec., 1998.

Carmody, *Killing them softly: inaccessible and Mahlo Cardinals* Mathematical Logic Quarterly, Vol

Easton, *Powers of regular cardinals.* Ann. Math. Logic 1, 139-178, MR 42 #4392, 1970.

Hamkins, *Forcing and Large cardinals*, based on CUNY set theory lecture 2004.

Hamkins, *Destruction or preservation: as you like it.* Ann. Pure Appl. Logic, vol. 91, iss. 2-3, pp. 191-229, 1998.

Hamkins, *Extensions with the approximation and cover properties have no new large cardinals.* Fundamenta Mathematicae 180(3):257-277, 2003.

References

Hamkins 5, *A multiverse perspective on the axiom of constructibility* Infinity and Truth, Vol. 25, 2013.

Hamkins, Shelah. *Superdestructibility: a dual to Laver's indestructibility.* Journal of Symbolic Logic, vol. 63, iss. 2, pp. 549-554, 1998.

Jech, *Set Theory*. Third Millenium Edition, Revised and Expanded, Springer Monographs in Mathematics, p. 120, 2002.
Jensen, Kunen, *Some combinatorial properties of L and V.* Unpublished, 1969.

Kanamori, *The Higher Infinite.* Springer Monographs in Mathematics, 2nd edition, 2003.

Magidor, *A Study on Identity Crises.* Ann. pure appl. logic, vol. 10, pp. 33-57, 1976.

Thank you!